3rd International Olympiad on Astronomy and Astrophysics

Theoretical Competition

Long Problems

Problem 16: High Altitude Projectile (45 points)

A projectile which initially is put on the surface of the Earth at the sea level is launched with the initial speed of $v_\circ = \sqrt{(GM/R)}$ and with the projecting angle (with respect to the local horizon) of $\theta = \frac{\pi}{6}$. M and R are the mass and radius of the Earth respectively. Ignore the air resistance and rotation of the Earth.

a) Show that the orbit of the projectile is an ellipse with a semi-major axis of $a = R$.
b) Calculate the highest altitude of the projectile with respect to the Earth surface (in the unit of the Earth radius).
c) What is the range of the projectile (distance between launching point and falling point)?
d) What is eccentricity (e) of the ellipse?
e) Find the flying time for the projectile.

Problem 17: Apparent number density of stars in the Galaxy (45 points)

Let us model the number density of stars in the disk of Milky Way Galaxy with a simple exponential function of $n = n_0 \exp\left(-\frac{r-R_0}{R_d}\right)$, where r represents the distance from the center of the Galaxy, R_0 is the distance of the Sun from the center of the Galaxy, R_d is the typical size of disk and n_0 is the stellar density of disk at the position of the Sun. An astronomer observes the center of the Galaxy within a small field of view. We take a particular type of Red giant stars (red clump) as the standard candles for the observation with approximately constant absolute magnitude of $M = -0.2$,

a) Considering a limiting magnitude of $m = 18$ for a telescope, calculate the maximum distance that telescope can detect the red clump stars. For simplicity we ignore the presence of interstellar medium so there is no extinction.
b) Assume an extinction of $0.70\, mag/kpc$ for the interstellar medium. Repeat the calculation as done in the part (a) and obtain a rough number for the maximum distance these red giant stars can be observed.
c) Give an expression for the number of these red giant stars per magnitude within a solid angle of Ω that we can observe with apparent magnitude in the range of m and $m + \Delta m$, (i.e. $\frac{\Delta N}{\Delta m}$). Red giant stars contribute f of overall stars. In this part assume no extinction in the interstellar medium as part (a).
Hint : the Tylor expansion of $y = \log_{10} x$ is :

$$y = y_0 + \frac{1}{ln10}\frac{x - x_0}{x}$$

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Solutions

Solution 16:

a) Total energy of the projectile is

$$E = \frac{1}{2}mv_{\circ}^2 - \frac{GMm}{R} = -\frac{GMm}{2R} < 0$$

$E < 0$ means that orbit might be ellipse or circle. As $\theta > 0$, the orbit is an ellipse.
Total energy for an ellipse is

$$E = -\frac{GMm}{2a}$$

Then

$$a = R$$

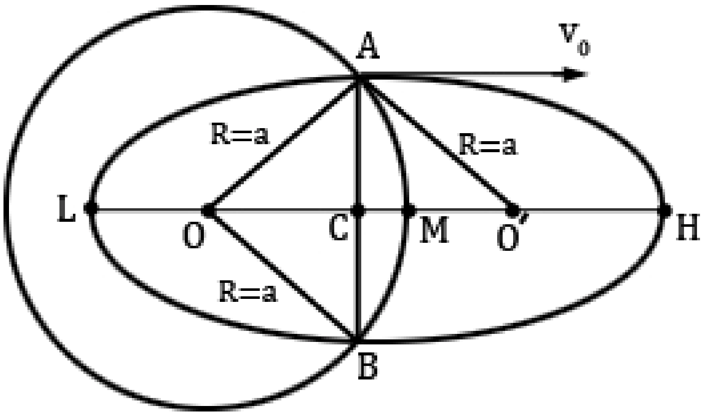

Figure (1)

b) In figure (1) we have

$$OA + O'A = 2a$$

$$O'A = a$$

In OAO' triangle it is obvious that

$$OC = CO'$$

Then C must be the center of the ellipse with the initial velocity vector $v_\circ$ parallel to the ellipse major-axis (LH).

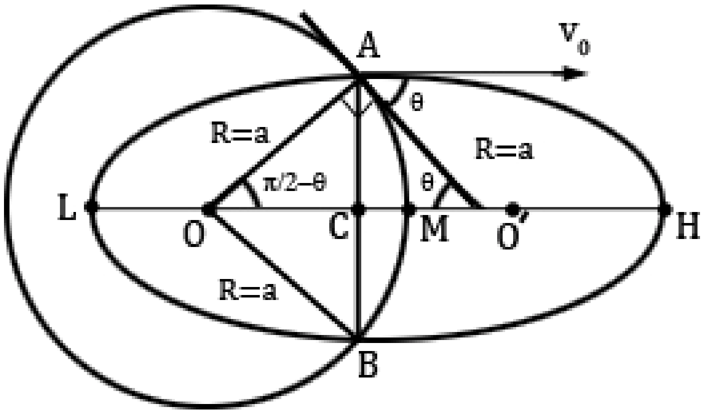

Figure (2)

In figure (2)

$$HM = CH - CM = a - (R - R\sin\theta) = R - R + R\sin\theta = R\sin\theta = \frac{\mathrm{R}}{2}$$

c) Range of the projectile is $\widehat{AB}$

$$\widehat{AB} = 2\left(\frac{\pi}{2} - \theta\right)R = (\pi - 2\theta)R = \frac{2\pi}{3}R$$

d) Start with ellipse equation in polar coordinates

$$r = \frac{a(1-e^2)}{1 + ecos\varphi}$$

For point A

$$R = \frac{R(1-e^2)}{1 - ecos(\frac{\pi}{2} + \theta)}$$

$$e = sin\theta = \frac{1}{2}$$

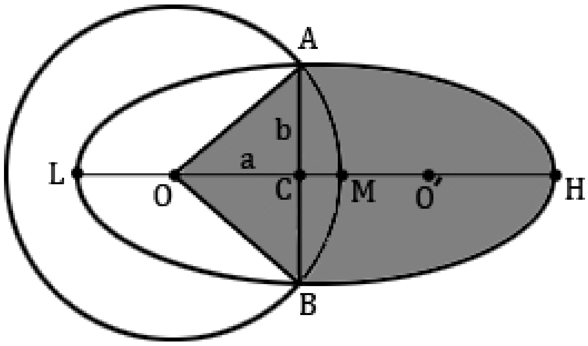

Figure 3

e) Using Kepler's second law

$$\frac{\Delta S}{S_0} = \frac{\Delta T}{T}$$

$$\Delta S = S_{AOBH} = S_{\Delta AOB} + \frac{S_0}{2}$$

$$= 2 \times \frac{\mathrm{bae}}{2} + \frac{\pi \mathrm{ab}}{2} = \mathrm{bae} + \frac{\pi \mathrm{ab}}{2}$$

$$\frac{\Delta S}{S} = \frac{\mathrm{bae} + \frac{\pi \mathrm{ab}}{2}}{\pi ab} = \frac{e + \frac{\pi}{2}}{\pi} = \frac{0.5 + \frac{\pi}{2}}{\pi}$$

Kepler's third law

$$T = \sqrt{\frac{4\pi^2 R^3}{GM}} = 84.5\ min$$

$$\Delta T = T \times \frac{0.5 + \frac{\pi}{2}}{\pi} = 55.7\ min$$

Solution 17:

a) Relation between the apparent and absolute magnitude is given by

$$m = M + 5log\left(\frac{d}{10}\right)$$

where d is in terms of parsec. Substituting $m = 18$ and $M = -0.2$, results in

$$d = 4.37 \times 10^4\ pc$$

b) Adding the term for the extinction, changes the magnitude distance relation as follows

$$m = M + 0.7x + 5\log(100x)$$

where x is given in terms of kilo parsec. To have a rough value for x, after substituting m and M, this equation reduces to

$$18.2 = 0.7x + 5\log(x)$$

To solve this equation, we examine

$$x = 5\,, 5.5\,, 6\,, 6.5$$

where the best value is obtained roughly $x \cong 6.1\ kpc$.

c) For a solid angle Ω, the number of observed red clump stars at the distance in the range of x and $x + \Delta x$ is given by

$$\Delta N = \Omega\, x^2\, n(x) f\, \Delta x$$

So the number of stars observed in Δx is given by

$$\frac{\Delta N}{\Delta x} = \Omega\, x^2\, n(x) f$$

From the relation between the distance and apparent magnitude we have

$$m_1 = M + 5log\left(\frac{x}{10}\right)$$

$$m_2 = M + 5log\left(\frac{x + \Delta x}{10}\right)$$

$$\Delta m = 5log\left(\frac{x + \Delta x}{x}\right)$$

$$\Delta m = 5log\left(1 + \frac{\Delta x}{x}\right)$$

$$\Delta m = \frac{5}{ln10} ln\left(1 + \frac{\Delta x}{x}\right)$$

$$\Delta m = \frac{5}{ln10}\left(\frac{\Delta x}{x}\right)$$

Replacing Δx with Δm, results in

$$\frac{\Delta N}{\Delta m} = \frac{\Delta N}{\Delta x} \times \frac{\Delta x}{\Delta m}$$

So the number of stars for a given magnitude is obtained by

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} n(x)x^3 f$$

Finally we substituting x in terms of apparent magnitude using $\mathrm{x} = 10^{\frac{\mathrm{m}-9.78}{5}}$.
In the case of no extinction, we are able to observe the Galaxy beyond the center. So $\frac{dN}{dm}$ has two terms in $x < R_0$ and $x > R_0$. The relation between x and r for these two cases are

$$x = R_0 - r \qquad x < R_0$$

and

$$x = R_0 + r \qquad x > R_0$$

So in general we can write $\frac{\Delta N}{\Delta m}$ as

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} n_0 \exp\left(\frac{10^{\frac{\mathrm{m}-9.78}{5}}}{R_d}\right) \times 10^{\frac{3(m-9.78)}{5}} f \qquad x < R_0$$

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} \mathrm{n_0} \exp\left(\frac{2\mathrm{R_0}}{R_d}\right) \exp\left(-\frac{10^{\frac{\mathrm{m}-9.78}{5}}}{R_d}\right) \times 10^{\frac{3(m-9.78)}{5}} f\ \Theta(x_0 - x) \quad x > R_0$$

where $\Theta(x)$ is the step function and $x_0 = 44.1\,kpc$ is the maximum observable distance.